A recurrence for the sequence $\{F_{F_n}, n \geq 0\}$

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1 Definitions.

The Fibonacci sequence we all love is defined by

$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}, \text{ for } n \geq 2. \tag{1}$$

This paper will address some interesting properties of the sequence

$$F_n^{(2)} = F_{F_n}, n \geq 0 \tag{2}$$

which begins {0, 1, 1, 1, 2, 5, 21, 233, 10946, 5702887, 139583862445...}.

Theorem 1.

$$\frac{F_{n+3}^{(2)}}{F_{n+2}^{(2)}} \sim \frac{F_{n+2}^{(2)}}{F_n^{(2)}}, \textit{ or } F_{n+3}^{(2)} F_n^{(2)} \sim {F_{n+2}^{(2)}}^2. \tag{3}$$

Proof. Recall Binet's formula

$$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}, \text{ where } \phi = \frac{1+\sqrt{5}}{2}, \psi = \frac{1-\sqrt{5}}{2}. \tag{4}$$

This gives

$$\begin{aligned} F_{n+3}^{(2)} F_n^{(2)} &= \frac{(\phi^{F_{n+3}} - \psi^{F_{n+3}})(\phi^{F_n} - \psi^{F_n})}{5} \\ &= \frac{(\phi^{F_{n+3}+F_n} + \psi^{F_{n+3}+F_n}) - \phi^{F_{n+3}}\psi^{F_n} - \phi^{F_n}\psi^{F_{n+3}}}{5} \\ &= \frac{(\phi^{F_{n+3}+F_n} + \psi^{F_{n+3}+F_n}) - (\phi\psi)^{F_n}(\phi^{F_{n+3}-F_n} + \psi^{F_{n+3}-F_n})}{5} \\ &= \frac{(\phi^{2F_{n+2}} + \psi^{2F_{n+2}}) - (\phi\psi)^{F_n}(\phi^{2F_{n+1}} + \psi^{2F_{n+1}})}{5} \end{aligned}$$

Since $\phi\psi = -1$, we have

$$F_{n+3}^{(2)} F_n^{(2)} = \frac{(\phi^{2F_{n+2}} + \psi^{2F_{n+2}}) - (-1)^{F_n}(\phi^{2F_{n+1}} + \psi^{2F_{n+1}})}{5} \tag{5}$$

Also,

$$\begin{aligned} {F_{n+2}^{(2)}}^2 &= \frac{(\phi^{F_{n+2}} - \psi^{F_{n+2}})(\phi^{F_{n+2}} - \psi^{F_{n+2}})}{5} \\ &= \frac{(\phi^{2F_{n+2}} + \psi^{2F_{n+2}}) - 2\psi^{F_{n+2}}\phi^{F_{n+2}}}{5} \\ &= \frac{(\phi^{2F_{n+2}} + \psi^{2F_{n+2}}) - 2(-1)^{F_{n+2}}}{5} \end{aligned} \tag{6}$$

We seek to prove

$$\lim_{n\to\infty} \frac{F_{n+3}^{(2)} F_n^{(2)}}{{F_{n+2}^{(2)}}^2} = 1$$

$$\lim_{n\to\infty} \frac{F_{n+3}^{(2)} F_n^{(2)}}{{F_{n+2}^{(2)}}^2} = \lim_{n\to\infty} \frac{\frac{(\phi^{2F_{n+2}}+\psi^{2F_{n+2}})-(-1)^{F_n}(\phi^{2F_{n+1}}+\psi^{2F_{n+1}})}{5}}{\frac{(\phi^{2F_{n+2}}+\psi^{2F_{n+2}})-2(-1)^{F_{n+2}}}{5}}$$

$$= \lim_{n\to\infty} \frac{(\phi^{2F_{n+2}} + \psi^{2F_{n+2}}) - (-1)^{F_n}(\phi^{2F_{n+1}} + \psi^{2F_{n+1}})}{(\phi^{2F_{n+2}} + \psi^{2F_{n+2}}) - 2(-1)^{F_{n+2}}}$$

As n gets large, $2 \ll (\phi^{2F_{n+2}} + \psi^{2F_{n+2}})$, so the $2(-1)^{F_{n+2}}$ term may be ignored:

$$\lim_{n\to\infty} \frac{(\phi^{2F_{n+2}} + \psi^{2F_{n+2}}) - (-1)^{F_n}(\phi^{2F_{n+1}} + \psi^{2F_{n+1}})}{(\phi^{2F_{n+2}} + \psi^{2F_{n+2}})}$$

$$= \lim_{n\to\infty} 1 - \frac{(-1)^{F_n}(\phi^{2F_{n+1}} + \psi^{2F_{n+1}})}{(\phi^{2F_{n+2}} + \psi^{2F_{n+2}})}$$

I will pause to make a note. Notice that if $F_n \equiv 1 \pmod 2$, or equivalently, $n \not\equiv 0 \pmod 3$, the quantity on the inside of the limit is a bit greater than 1; otherwise, it is a bit smaller.

Now what we want is to show that

$$\lim_{n\to\infty} \frac{(-1)^{F_n}(\phi^{2F_{n+1}} + \psi^{2F_{n+1}})}{(\phi^{2F_{n+2}} + \psi^{2F_{n+2}})} = 0$$

We may take the absolute value inside the limit to rid ourselves of the $(-1)^{F_n}$ term. So it is sufficient to show that

$$\lim_{n\to\infty} \frac{(\phi^{2F_{n+1}} + \psi^{2F_{n+1}})}{(\phi^{2F_{n+2}} + \psi^{2F_{n+2}})} = 0$$

As $n \to \infty$, $\psi^n \to 0$, so we are left with the easy task of showing

$$\lim_{n\to\infty} \frac{\phi^{2F_{n+1}}}{\phi^{2F_{n+2}}} = 0$$

$\frac{\phi^{2F_{n+1}}}{\phi^{2F_{n+2}}} = \phi^{2(F_{n+1}-F_{n+2})} = \phi^{-2F_n} \to 0$, so our work is done. ■

This sequence converges very quickly. Indeed, $\frac{F_{12}^{(2)} F_9^{(2)}}{{F_{11}^{(2)}}^2} \approx 1 - (10^{-14})$.

Theorem 2. *A recurrence for* $F_n^{(2)}$.

$$F_n^{(2)} = \frac{{F_{n-1}^{(2)}}^2 - (-1)^{F_{n-3}} F_{n-4}^{(2)} F_{n-1}^{(2)} - (-1)^{F_{n-2}} {F_{n-3}^{(2)}}^2}{F_{n-3}^{(2)}}, n \geq 4.$$

Proof. We rewrite the right side using Binet's formula:

$$\frac{(\phi^{F_{n-1}}-\psi^{F_{n-1}})^2-(-1)^{F_{n-3}}(\phi^{F_{n-4}}-\psi^{F_{n-4}})(\phi^{F_{n-1}}-\psi^{F_{n-1}})-(-1)^{F_{n-2}}(\phi^{F_{n-3}}-\psi^{F_{n-3}})^2}{\sqrt{5}(\phi^{F_{n-3}}-\psi^{F_{n-3}})}$$

$$=\frac{(\phi^{F_{n-1}}-\psi^{F_{n-1}})^2-(\phi\psi)^{F_{n-3}}(\phi^{F_{n-4}}-\psi^{F_{n-4}})(\phi^{F_{n-1}}-\psi^{F_{n-1}})-(\phi\psi)^{F_{n-2}}(\phi^{F_{n-3}}-\psi^{F_{n-3}})^2}{\sqrt{5}(\phi^{F_{n-3}}-\psi^{F_{n-3}})}$$

Now, we will simplify the numerator:

$$\begin{aligned}&(\phi^{F_{n-1}}-\psi^{F_{n-1}})^2-(\phi\psi)^{F_{n-3}}(\phi^{F_{n-4}}-\psi^{F_{n-4}})(\phi^{F_{n-1}}-\psi^{F_{n-1}})-(\phi\psi)^{F_{n-2}}(\phi^{F_{n-3}}-\psi^{F_{n-3}})^2\\ &=\phi^{2F_{n-1}}-\phi^{F_{n-4}+F_{n-3}+F_{n-1}}\psi^{F_{n-3}}+\phi^{F_{n-3}+F_{n-1}}\psi^{F_{n-4}+F_{n-3}}-\phi^{2F_{n-3}+F_{n-2}}\psi^{F_{n-2}}+\\ &2\phi^{F_{n-3}+F_{n-2}}\psi^{F_{n-3}+F_{n-2}}-\phi^{F_{n-2}}\psi^{2F_{n-3}+F_{n-2}}-2\phi^{F_{n-1}}\psi^{F_{n-1}}+\psi^{2F_{n-1}}+\\ &\phi^{F_{n-4}+F_{n-3}}\psi^{F_{n-3}+F_{n-1}}-\phi^{F_{n-3}}\psi^{F_{n-4}+F_{n-3}+F_{n-1}}\end{aligned}$$

Using $F_{n-4}+F_{n-3}+F_{n-1}=F_n$, we may write:

$$\phi^{2F_{n-1}}-\phi^{F_n}\psi^{F_{n-3}}+\phi^{F_{n-3}+F_{n-1}}\psi^{F_{n-4}+F_{n-3}}-\phi^{2F_{n-3}+F_{n-2}}\psi^{F_{n-2}}+2\phi^{F_{n-3}+F_{n-2}}\psi^{F_{n-3}+F_{n-2}}-$$
$$\phi^{F_{n-2}}\psi^{2F_{n-3}+F_{n-2}}-2\phi^{F_{n-1}}\psi^{F_{n-1}}+\psi^{2F_{n-1}}+\phi^{F_{n-4}+F_{n-3}}\psi^{F_{n-3}+F_{n-1}}-\phi^{F_{n-3}}\psi^{F_n}$$

Since $2F_{n-1}=F_n+F_{n-3}$, this is equal to

$$\phi^{F_n+F_{n-3}}-\phi^{F_n}\psi^{F_{n-3}}+\phi^{F_{n-3}+F_{n-1}}\psi^{F_{n-4}+F_{n-3}}-\phi^{2F_{n-3}+F_{n-2}}\psi^{F_{n-2}}+2\phi^{F_{n-3}+F_{n-2}}\psi^{F_{n-3}+F_{n-2}}-$$
$$\phi^{F_{n-2}}\psi^{2F_{n-3}+F_{n-2}}-2\phi^{F_{n-1}}\psi^{F_{n-1}}+\psi^{F_n+F_{n-3}}+\phi^{F_{n-4}+F_{n-3}}\psi^{F_{n-3}+F_{n-1}}-\phi^{F_{n-3}}\psi^{F_n}$$

Now we use $F_{n-4}+F_{n-3}=F_{n-2}$:

$$\phi^{F_n+F_{n-3}}-\phi^{F_n}\psi^{F_{n-3}}+\phi^{F_{n-3}+F_{n-1}}\psi^{F_{n-2}}-\phi^{2F_{n-3}+F_{n-2}}\psi^{F_{n-2}}+2\phi^{F_{n-3}+F_{n-2}}\psi^{F_{n-3}+F_{n-2}}-$$
$$\phi^{F_{n-2}}\psi^{2F_{n-3}+F_{n-2}}-2\phi^{F_{n-1}}\psi^{F_{n-1}}+\psi^{F_n+F_{n-3}}+\phi^{F_{n-2}}\psi^{F_{n-3}+F_{n-1}}-\phi^{F_{n-3}}\psi^{F_n}$$

Since $2F_{n-3}+F_{n-2}=F_{n-3}+F_{n-1}$:

$$\begin{aligned}&\phi^{F_n+F_{n-3}}-\phi^{F_n}\psi^{F_{n-3}}+\phi^{F_{n-3}+F_{n-1}}\psi^{F_{n-2}}-\phi^{F_{n-3}+F_{n-1}}\psi^{F_{n-2}}+2\phi^{F_{n-3}+F_{n-2}}\psi^{F_{n-3}+F_{n-2}}-\\ &\phi^{F_{n-2}}\psi^{F_{n-3}+F_{n-1}}-2\phi^{F_{n-1}}\psi^{F_{n-1}}+\psi^{F_n+F_{n-3}}+\phi^{F_{n-2}}\psi^{F_{n-3}+F_{n-1}}-\phi^{F_{n-3}}\psi^{F_n}\\ &=\phi^{F_n+F_{n-3}}-\phi^{F_n}\psi^{F_{n-3}}+2\phi^{F_{n-3}+F_{n-2}}\psi^{F_{n-3}+F_{n-2}}-2\phi^{F_{n-1}}\psi^{F_{n-1}}+\\ &\psi^{F_n+F_{n-3}}-\phi^{F_{n-3}}\psi^{F_n}\end{aligned}$$

Finally we use $F_{n-3}+F_{n-2}=F_{n-1}$:

$$\begin{aligned}&\phi^{F_n+F_{n-3}}-\phi^{F_n}\psi^{F_{n-3}}+2\phi^{F_{n-1}}\psi^{F_{n-1}}-2\phi^{F_{n-1}}\psi^{F_{n-1}}+\psi^{F_n+F_{n-3}}-\phi^{F_{n-3}}\psi^{F_n}\\ &=\phi^{F_n+F_{n-3}}-\phi^{F_n}\psi^{F_{n-3}}+\psi^{F_n+F_{n-3}}+-\phi^{F_{n-3}}\psi^{F_n}\\ &=(\phi^{F_{n-3}}-\psi^{F_{n-3}})(\phi^{F_n}-\psi^{F_n})\end{aligned}$$

With the numerator simplified, the right side of the equation becomes

$$\frac{(\phi^{F_{n-3}}-\psi^{F_{n-3}})(\phi^{F_n}-\psi^{F_n})}{(\phi^{F_{n-3}}-\psi^{F_{n-3}})\sqrt{5}}=\frac{(\phi^{F_n}-\psi^{F_n})}{\sqrt{5}}=F_{F_n}.\blacksquare$$